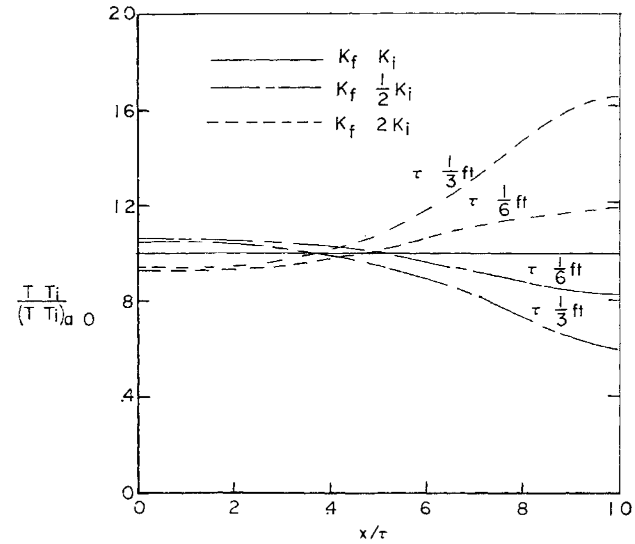

Fig 2 Comparison of tempeiatures in skins having constant and nonconstant thermal conductivities

increasing to twice its initial value Figure 2 shows a comparison of the temperature in a skin having a nonconstant thermal conductivity to one for which the thermal conductivity is constant The curves in Fig 2 show a slight relief in temperature up to about the middle of the skin for the case of the thermal conductivity increasing with depth However, for the back half of the skin, this variation in thermal conductivity shows a rapid increase in temperature On the other hand, there is a slight temperature rise in the first half of the skin for the case of decreasing thermal conductivity with an appreciable decrease in temperature for the back half of the skin for this variation in thermal conductivity

For these figures an entry velocity of 20,000 fps at 400,000 ft and an entry angle of $-20°$ was used The initial value of the thermal conductivity used was 0 548 Btu/ft-sec-°F which corresponds to electrolytic copper at 1000°F The re sults shown are time independent after about 10 sec

Refeiences

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Nth Order Solutions to Certain Thrust Integrals

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An exact solution is presented to certain thrust integrals which previously have been solved with a linearization technique The solution is achieved by identifying two well known integral forms that aie common to all components of the thrust integials and whose series solutions conveige for all values of the independent vaiiable

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Nomenclatuie

A, **B**	=	thrust integral defined specifically by Eq (2)
a	=	constant involving summations of $\dot\theta$, $\dot\psi$ and ω
f	=	function of sinat or cosat
F	=	thrust vector
I	=	integral function defined specifically by Eq (10)
J	=	integral function defined specifically by Eq (9)
i, **j**, **k**	=	mutually orthogonal ineitial unit vectors
m	=	vehicle mass
r	=	central body radius vector to vehicle
t	=	time
T	=	vehicle burnup time, $(m_0/\dot m)$
x, y, z	=	inertial Cartesian coordinate system
θ	=	coangle between thrust vector and z axis
μ	=	central body gravitational constant
ν	=	value of v at time $= \tau$
τ	=	burning time
v	=	dummy variable
ψ	=	angle between x axis and piojection of **F** in x-y plane
ω	=	mean motion $= \{\mu/[\frac{1}{2}(r_0 + r_t)]^3\}^{1/2}$

Subscripts

0	=	at time zero
s	=	function involving a sine
c	=	function involving a cosine
i	=	function involving the ith form of the constant a
x, y, z,	=	in the direction of **i** **j** or **k**

Intioduction

REFERENCE 1 presents a solution to the vector differential equation of motion in a central force field:

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{\mu}{r^3}\mathbf{r} + \frac{\mathbf{F}}{m} \tag{1}$$

with the assumption that the change in (μ/r^3) is small with respect to the thrust acceleration vector This solution, though exact within the assumption as stated, requires the evaluation of certain thrust integrals **A** and **B** which have the following form:

$$\mathbf{A} = \frac{1}{\omega}\int_0^\tau \frac{\mathbf{F}}{m}\cos\omega t\, dt$$
$$\mathbf{B} = \frac{1}{\omega}\int_0^\tau \frac{\mathbf{F}}{m}\sin\omega t\, dt \tag{2}$$

Reference 2 develops a first order evaluation of **A** and **B** from the following formulation for **F**:

$$F_x(t) = F(t)\cos(\theta_0 + \dot\theta t)\cos(\psi_0 + \dot\psi t)$$
$$F_y(t) = F(t)\cos(\theta_0 + \dot\theta t)\sin(\psi_0 + \dot\psi t) \tag{3}$$
$$F\ (t) = F(t)\sin(\theta_0 + \dot\theta t)$$

and $\mathbf{F} = \mathbf{i}F_x + \mathbf{j}F_y - \mathbf{k}F$ where **i**, **j**, and **k** are mutually orthogonal inertial unit vectors Although the first-order solution[2] was adequate for the expressed intent, it is not necessary to restrict oneself to first order solutions for **A** and **B** To the contrary, nth order (i e, exact) solutions may be achieved and it is the purpose of this note to present these

Theory

Consider first the component of **A** in the $-\mathbf{k}$ direction:

$$-\omega A_z = \int_0^\tau \frac{F_z}{m}\cos\omega t\, dt \tag{4}$$

Also note that[1]

$$\omega\dot{\mathbf{A}} = (\mathbf{F}/m)\cos\omega t \qquad m = m_0 - \dot m t \qquad T = m_0/\dot m \tag{5}$$

Performing appropriate substitutions for F and m, obtain

$$-\omega \dot{A} = \frac{F}{m_0} \frac{\sin(\theta_0 + \dot{\theta}t)}{1 - t/T} \cos\omega t$$

which may be expanded to the form

$$-\omega \dot{A}_z = \frac{F}{m_0} \frac{\sin\theta_0 \cos\dot{\theta}t \cos\omega t + \cos\theta_0 \sin\dot{\theta}t \cos\omega t}{1 - t/T} \quad (6)$$

Consider the first term on the right of the equality sign It can be shown that

$$\cos\dot{\theta}t \cos\omega t \equiv \tfrac{1}{2}[\cos(\dot{\theta} + \omega)t + \cos(\dot{\theta} - \omega)t] \quad (7)$$

A similar identity is applicable to the second term of Eq (6):

$$\sin\dot{\theta}t \cos\omega t \equiv \tfrac{1}{2}[\sin(\dot{\theta} + \omega)t + \sin(\dot{\theta} - \omega)t] \quad (8)$$

It is apparent that the expression for $\dot{A}_z$ can be reduced to an expression of the form

$$\dot{A} = C_1 \frac{\sin at}{1 - t/T} + C_2 \frac{\cos bt}{1 - t/T}$$

By an extension of this argument to the other components of **A** and **B**, it will be shown that this general form can be achieved for these components as well Therefore, the integrals that are to be evaluated are of the form

$$J_{i} \equiv \frac{1}{T} \int_0^{\tau} \frac{\sin a_i t}{1 - t/T} dt \qquad (i = 1, 2, \quad , 6)$$

$$J_{i} \equiv \frac{1}{T} \int_0^{\tau} \frac{\cos a_i t}{1 - t/T} dt \quad (9)$$

This evaluation is most easily achieved by introducing a change of variable Define the new variable v_i such that $t = T - v_i/a_i$ and $dt = -dv_i/a_i$ The integrals in question now appear as

$$J_{i} = -\sin a_i T \int_{a_i T}^{\nu_i} \frac{\cos v_i}{v_i} dv_i + \cos a_i T \int_{a_i T}^{\iota} \frac{\sin v_i}{v_i} dv_i$$

$$J_{i} = -\cos a_i T \int_{a_i T}^{\nu_i} \frac{\cos v_i}{v_i} dv_i - \sin a_i T \int_{a_i T}^{\nu_i} \frac{\sin v_i}{v_i} dv_i$$

where $\nu_i = a_i(T - \tau)$

The solutions to the indefinite integrals corresponding to those above are now written as infinite series[3]:

$$I(\chi) = \int \frac{\cos\chi}{\chi} d\chi = \ln\chi + \sum_{n=1}^{\infty} (-1)^n \frac{\chi^{2n}}{2n(2n)!}$$

$$I(\chi) = \int \frac{\sin\chi}{\chi} d\chi = \chi + \sum_{n=1}^{\infty} (-1)^n \frac{\chi^{2n+1}}{(2n+1)(2n+1)!} \quad (10)$$

Rewriting the J function in terms of the I function, the following solutions are obtained for the general form of the integrals that appear in the thrust integrals **A** and **B**:

$$J_{i} = -\sin a_i T[I(\nu_i) - I(a_i T)] + \cos a_i T[I(\nu_i) - I(a_i T)]$$

$$J_{ci} = -\cos a_i T[I(\nu_i) - I_c(a_i T)] - \sin a_i T[I(\nu_i) - I(a_i T)] \quad (11)$$

With Eqs (11) stated, it is now possible to elaborate on the statement of the complete solution to the components of **A** and **B** Consider the following equality obtained from Eqs (3) and (5):

$$\frac{\omega}{V_j}\begin{bmatrix} \dot{A}_x \\ \dot{A}_y \\ \dot{A} \\ \dot{B}_x \\ \dot{B}_y \\ \dot{B} \end{bmatrix} = \begin{bmatrix} \cos(\theta_0 + \dot{\theta}t)\cos(\psi_0 + \dot{\psi}t)\cos\omega t \\ \cos(\theta_0 + \dot{\theta}t)\sin(\psi_0 + \dot{\psi}t)\cos\omega t \\ -\sin(\theta_0 + \dot{\theta}t)\cos\omega t \\ \cos(\theta_0 + \dot{\theta}t)\cos(\psi_0 + \dot{\psi}t)\sin\omega t \\ \cos(\theta_0 + \dot{\theta}t)\sin(\psi_0 + \dot{\psi}t)\sin\omega t \\ -\sin(\theta_0 + \dot{\theta}t)\sin\omega t \end{bmatrix} \frac{1}{T - t} \quad (12)$$

where $V_j = F/\dot{m}$

The functions on the right of the equality sign are to be rearranged, just as Eq (6) was earlier, to obtain the elements in the form of the general integral just described through the statement of Eq (9) In addition to the two identities given by Eqs (7) and (8), a third identity is required:

$$\sin\dot{\theta}t \sin\omega t \equiv -\tfrac{1}{2}[\cos(\dot{\theta} + \omega)t - \cos(\dot{\theta} - \omega)t] \quad (13)$$

The algebraic manipulation is straightforward, but tedious; only the result is given below:

$$\frac{\omega}{V_j}\begin{bmatrix} \dot{A}_x \\ \dot{A}_y \\ \dot{A} \\ \dot{B}_x \\ \dot{B}_y \\ \dot{B} \end{bmatrix} \equiv \begin{bmatrix} f_{3} + f_{4} & (f_{5} + f_{6}) & -(f_{3} + f_{s4}) & -(f_{5} + f_{6}) & 0 & 0 \\ f_{s3} + f_{s4} & -(f_{5} + f_{6}) & (f_{3} + f_{c4}) & -(f_{c5} + f_{6}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(f_{1} + f_{2}) & 2(f_{1} + f_{2}) \\ f_{3} + f_{4} & (f_{5} + f_{6}) & (f_{3} - f_{4}) & (f_{5} - f_{6}) & 0 & 0 \\ -f_{3} + f_{4} & (f_{5} - f_{6}) & (f_{5} + f_{6}) & -(f_{3} + f_{s4}) & 0 & 0 \\ 0 & 0 & 0 & 0 & -2(f_{1} - f_{2}) & 2(f_{1} + f_{2}) \end{bmatrix} \begin{bmatrix} \cos(\theta_0 + \psi_0) \\ \cos(\theta_0 - \psi_0) \\ \sin(\theta_0 + \psi_0) \\ \sin(\theta_0 - \psi_0) \\ \cos\theta_0 \\ \sin\theta_0 \end{bmatrix} \left(\frac{1}{T - t}\right) \quad (14)$$

where

$$\begin{aligned} f_{i} &= \tfrac{1}{4}\cos a_i t \qquad (i = 1, 2, \quad , 6) \\ f_{i} &= \tfrac{1}{4}\sin a_i t \\ a_1 &= \dot{\theta} + \omega \\ a_2 &= \dot{\theta} - \omega \\ a_3 &= \dot{\theta} + \dot{\psi} + \omega \\ a_4 &= \dot{\theta} + \dot{\psi} - \omega \\ a_5 &= \dot{\theta} - \dot{\psi} + \omega \\ a_6 &= \dot{\theta} - \dot{\psi} - \omega \end{aligned}$$

It remains now only to integrate between the limits 0 and τ to complete the solution for **A** and **B** This follows immediately using the notation developed previously:

$$4\frac{\omega}{V_j}\begin{bmatrix} A_x \\ A_y \\ A \\ B_x \\ B_y \\ B \end{bmatrix} = \begin{bmatrix} J_{3} + J_{c4} & (J_{5} + J_{6}) & -(J_{3} + J_{4}) & -(J_{5} + J_{6}) & 0 & 0 \\ J_{3} + J_{s4} & -(J_{5} + J_{6}) & (J_{3} + J_{4}) & -(J_{c5} + J_{c6}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(J_{1} + J_{2}) & 2(J_{c1} + J_{2}) \\ J_{s3} + J_{4} & (J_{s5} + J_{6}) & (J_{3} - J_{4}) & (J_{5} - J_{6}) & 0 & 0 \\ -J_{c3} + J_{4} & (J_{5} - J_{c6}) & (J_{5} + J_{6}) & -(J_{3} + J_{4}) & 0 & 0 \\ 0 & 0 & 0 & 0 & -2(J_{c1} - J_{2}) & 2(J_{1} + J_{2}) \end{bmatrix} \begin{bmatrix} \cos(\theta_0 + \psi_0) \\ \cos(\theta_0 - \psi_0) \\ \sin(\theta_0 + \psi_0) \\ \sin(\theta_0 - \psi_0) \\ \cos\theta_0 \\ \sin\theta_0 \end{bmatrix} \quad (15)$$

Conclusion

The development presented in this note represents one exact solution to the time dependent thrust integrals which are required in the solution of the differential equation for a particle in a uniform central force field The time dependent form of the thrust vector used in this solution is commonly used for trajectory path control and launch guidance

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